Asymptotics for a discrete-time risk model with Gamma-like insurance risks

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Abstract

Consider a discrete-time insurance risk model with insurance and financial risks. Within period i, the net insurance loss is denoted by X_i and the stochastic discount factor over the same time period is denoted by Y_i. Assume that $\{X_i,\ i \geq 1\}$ form a sequence of independent and identically distributed real-valued random variables with common distribution F; $\{Y_i,\ i \geq 1\}$ are another sequence of independent and identically distributed positive random variables with common distribution G; and the two sequences are mutually independent. Under the assumptions that F is Gamma-like tailed and G has a finite upper endpoint, we derive some precise formulas for the tail probability of the present value of aggregate net losses and the finite-time and infinite-time ruin probabilities. As an extension, a dependent risk model is considered, where each random pair of the net loss and the discount factor follows a bivariate Sarmanov distribution.

Keywords: Asymptotics; Gamma-like tail; insurance and financial risks; finite-time and infinite-time ruin probabilities; stochastic discounted value of aggregate net losses; Sarmanov distribution

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1 Introduction and preliminaries

Consider a discrete-time risk model, where, for every $i \geq 1$, an insurer's net loss (the aggregate claim amount minus the total premium income) within period i is denoted by a real-valued random variable (r.v.) X_i; the stochastic discount factor (the reciprocal of the stochastic return rate) over the same time period is denoted by a positive r.v. Y_i; and $\{(X_i, Y_i),\ i \geq 1\}$ form a sequence of independent and identically distributed (i.i.d.) random vectors with marginal distributions F and G, respectively. In the terminology of [12], $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ represent the corresponding insurance risks and financial risks, respectively. In this framework, the stochastic discounted value of aggregate net losses can be specified as

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j,\ n \geq 1, \tag{1.1}$$

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with their maxima

$$M_n = \max_{0\le k\le n} S_k,\ n\ge 1, \quad M_\infty = \max_{k\ge 0} S_k.$$

Clearly, M_n is non-decreasing in n and

$$0 \le M_n \le \sum_{i=1}^{n} \max\{X_i, 0\} \prod_{j=1}^{i} Y_j. \tag{1.2}$$

It is well known that the right-hand side of (1.2) converges almost surely (a.s.) if $-\infty \le \mathbb{E}\ln Y_1 < 0$ and $\mathbb{E}\ln\max\{X_1, 1\} < \infty$, see Theorem 1.6 in [17] and Theorem 1 in [2]. Therefore, M_n converges a.s. to its maximum M_∞, which has a proper distribution function on $[0,\infty)$. In this paper we are interested in the asymptotic behavior of the tail probabilities $\mathbb{P}(S_n > x)$, $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$ as $x \to \infty$. We remark that $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$ coincide with the finite-time ruin probability within period n and the infinite-time ruin probability, respectively, when $x \ge 0$ is interpreted as the initial wealth of the insurer.

In such a discrete-time risk model, under independence or some certain dependence assumptions imposed on X_i's and Y_i's, the asymptotic tail behavior of S_n, M_n and M_∞ has been extensively investigated by many researchers. For some recent findings in the independent risk model, one can be referred to [14], [15], [10], [13], [9] and [8], among others. In this paper, we restrict the insurance risks to have a Gamma-like tail.

Throughout the paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 1$; write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \le 1$; and write $a(x) = o(b(x))$ if $\lim_{x\to\infty} a(x)/b(x) = 0$. For two real-valued numbers x and y, denote by $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and denote the positive part of x by $x_+ = x \vee 0$. The indicator function of an event A is denoted by $\mathbf{1}_A$.

A distribution F on $\mathbb{R}$ is said to have a Gamma-like tail with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ if there exists a slowly function $l(\cdot) : (0,\infty) \mapsto (0,\infty)$ such that

$$\overline{F}(x) = 1 - F(x) \sim l(x)x^{\alpha-1}e^{-\gamma x}. \tag{1.3}$$

A canonical example of the Gamma-like distribution with parameters $\alpha > 0$, $\gamma > 0$ is the Gamma distribution with the corresponding parameters, i.e.,

$$\overline{F}(x) = \frac{\gamma^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\gamma y}\mathrm{d}y,\ x > 0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function. More details on the distributions with Gamma-like tails can be found in [7], which studied the asymptotic tail behavior of the reinsured amounts under ECOMOR (excédent du coût moyen relatif) and LCR (largest claims reinsurance) reinsurance treaties. A distribution F on $\mathbb{R}$ is said to belong to the class $\mathscr{L}(\gamma)$ with $\gamma \ge 0$, if for any $y \in \mathbb{R}$, $\overline{F}(x - y) \sim e^{\gamma y}\overline{F}(x)$. If $\gamma = 0$, the classe $\mathscr{L}(0)$ consists all long-tailed distributions, which are heavy-tailed. If $\gamma > 0$, then all distributions in the class $\mathscr{L}(\gamma)$ are light-tailed. In this case, a class larger than the generalized exponential class $\mathscr{L}(\gamma)$, is that of rapidly varying tailed distributions, denoted by $\mathscr{R}_{-\infty}$. Clearly, if a distribution F has a Gamma-like tail with shape parameter $\alpha > 0$ and rate parameter $\gamma > 0$, then $F \in \mathscr{L}(\gamma) \subset \mathscr{R}_{-\infty}$.

In the case of heavy-tailed insurance risks, there has been a vast amount of literature. If we denote the product $\prod_{j=1}^{i} Y_j$ in (1.1) by a weight r.v. Θ_i, then the investigation on $\mathbb{P}(S_n > x)$, $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$ boils down to the study of the asymptotics for the tail probabilities of randomly weighted sums and their maximum. In the presence of subexponential insurance risks, [16] established the asymptotic formula

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x) \sim \sum_{i=1}^{n} \mathbb{P}(\Theta_i X_i > x) = \sum_{i=1}^{n} \mathbb{P}\Big(X_i \prod_{j=1}^{i} Y_j > x\Big)$$

holds for each fixed $n \geq 1$, under the conditions that the random weights $\{\Theta_i,\ 1 \leq i \leq n\}$ are nonnegative, not degenerate at 0, bounded above, and arbitrarily dependent on each other, but independent of $\{X_i,\ i \geq 1\}$.

In the present paper we aim to investigate the asymptotic tail behavior of S_n, M_n and M_∞, under the assumptions that the insurance risks X_i's have a common Gamma-like tailed distribution F, and the financial risks Y_i's have have a common distribution G with finite upper endpoint

$$y_* = y_*(G) = \sup\{y : G(y) < 1\} < \infty. \tag{1.4}$$

We remark that Theorem 4.3 in [15] and Theorem 2.2 in [13] made some similar investigation in this direction, and under the condition that Y_1 has a finite upper endpoint y_*, they derived many important and valuable results in the case X_1 has a convolution-equivalent or rapidly varying tail. Precisely speaking, consider a discrete-time risk model, where $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are two sequences of i.i.d. r.v.s with common distributions F and G, respectively, and the two sequences are mutually independent. Assume that the distribution G has a finite upper endpoint y_*. If $F \in \mathscr{R}_{-\infty}$ and $y_* > 1$ with $p_* = \mathbb{P}(Y_1 = y_*) > 0$, then for each fixed $n \geq 1$,

$$\mathbb{P}(M_n > x) \sim p_*^n \mathbb{P}\Big(\sum_{i=1}^{n} y_*^i X_i > x\Big). \tag{1.5}$$

Motivated by [15] and [13], in the present paper we investigate the tail asymptotics for S_n and M_n in the cases of $0 < y_* < 1$, $y_* > 1$ and $y_* = 1$, respectively, when F is Gamma-like tailed. Our obtained results do not need the restriction $p_* > 0$. Due to the concrete form of F, our obtained result in the case $y_* > 1$ presents a more accurate formula than (1.5). Further, we also consider the asymptotics behavior of $\mathbb{P}(M_\infty > x)$ in the case $0 < y_* < 1$, which, together with the asymptotic formula for $\mathbb{P}(M_n > x)$, leads to a uniform result for both finite-time and infinite-time ruin probabilities. In addition, an extension that incorporate a certain dependence structure into the model is considered.

The rest of this paper is organized as follows. Section 2 presents the main results of the present paper and Section 3 provides an extension for a dependent discrete-time risk model where a certain dependence is taken into account between insurance and financial risks. All the proofs are displayed in Section 4.

2 Main results

In this section, we restrict ourselves to the standard framework in which $\{X_i,\ i \geq 1\}$ form a sequence of i.i.d. real-valued r.v.s with common distribution F; $\{Y_i,\ i \geq 1\}$ form another sequence of i.i.d. positive r.v.s with common distribution G; and $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are mutually independent.

Now we state the main results, in which we denote $\psi_n(x) = p_* \mathbb{E}\big(e^{\gamma M_{n-1}}\big) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}$, $1 \leq n \leq \infty$. The first theorem derives some asymptotics for the finite-time ruin probability and the tail probability of the present value of the total net losses up to a finite time.

Theorem 2.1. *Consider the discrete-time risk model defined above. Assume that F is Gamma-like tailed with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ defined in* (1.3)*, and G has a finite upper endpoint y_* defined in* (1.4)*. Denote $K = p_*/y_*^{\alpha-1}$.*

(1) If $0 < y_* < 1$, then for each fixed $n \geq 1$, $\mathbb{E}\big(e^{\gamma S_{n-1}}\big) < \infty$, $\mathbb{E}\big(e^{\gamma M_{n-1}}\big) < \infty$, and

$$\mathbb{P}(S_n > x) \sim K\mathbb{E}\big(e^{\gamma S_{n-1}}\big)l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}, \quad \mathbb{P}(M_n > x) \sim \psi_n(x). \tag{2.1}$$

(2) If $y_* = 1$, then for each fixed $n \geq 1$,

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x) \sim \frac{K^n \gamma^{n-1}(\Gamma(\alpha))^n}{\Gamma(n\alpha)}(l(x))^n x^{n\alpha-1}e^{-\gamma x}. \tag{2.2}$$

(3) If $y_* > 1$, then for each fixed $n \geq 1$,

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x) \sim K^n \prod_{i=1}^{n-1} \mathbb{E}\big(e^{\frac{\gamma X_1}{y_*^i}}\big)l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*^n}}, \tag{2.3}$$

where $\prod_{i=1}^{0}(\cdot) = 1$ by convention.

We note that in Theorem 2.1 (1) and (2), the assumption $y_* \leq 1$ means that the insurer invests all his surplus into a risk-free asset and then he receives nonnegative stochastic returns. Theorem 2.1 (3) considers the case $y_* > 1$, which allows the insurer to make both risk-free and risky investments. Although [15] derived a general result than relation (2.3) to some extent, we establish a more precise formula for both $\mathbb{P}(M_n > x)$ and $\mathbb{P}(S_n > x)$ due to the concrete form of F, and without the restriction $p_* > 0$.

Our next result below shows that the second asymptotic relation of (2.1) for the finite-time probability is uniform over all positive integers $\{n \geq 1\}$ in the case $0 < y_* < 1$, which implies the uniform asymptotics for both finite-time and infinite-time ruin probabilities.

Theorem 2.2. *Under the conditions of Theorem 2.1, If $0 < y_* < 1$, then it holds that*

$$\mathbb{P}(M_\infty > x) \sim \psi_\infty(x); \tag{2.4}$$

further, the second relation of (2.1) *holds uniformly for all $n \geq 1$. That is*

$$\lim_{x\to\infty} \sup_{n\geq 1} \left| \frac{\mathbb{P}(M_n > x)}{\psi_n(x)} - 1 \right| = 0.$$

3 An extension

Undoubtedly, the assumption of complete independence on the two sequences of $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ is far unrealistic and of less practical relevance, though often appearing in the literature. In this section, we provide an extension for our main results in Section 2, by incorporating a certain dependence structure into the risk model. As done in [3], we assume that $\{(X_i, Y_i),\ i \geq 1\}$ form a sequence of i.i.d. random vectors whose components are however dependent. We use a bivariate Sarmanov distribution to model the dependence structure of (X_1, Y_1). More generally, [18] proposed a larger class of absolutely continuous (AC) product distributions, whose dependence structure is introduced by restricting (X_1, Y_1) such that its joint distribution is absolutely continuous with respect to the product distribution FG, i.e.,

$$\mathbb{P}(X_1 \leq x, Y_1 \leq y) = \int_{-\infty}^{x}\int_{0}^{y} \eta(u,v)F(\mathrm{d}u)G(\mathrm{d}v),\ x \in \mathbb{R},\ y \in (0, y_*],$$

where $\eta(\cdot,\cdot)$ is a nonnegative measurable function. Then, a bivariate Sarmanov distribution we consider is obtained when $\eta(x,y)=1+\theta\phi_1(x)\phi_2(y)$, where ϕ_1 and ϕ_2 are two given real-valued kernel functions and θ is a real parameter satisfying

$$\mathbb{E}\phi_1(X_1)=\mathbb{E}\phi_2(Y_1)=0,$$

and

$$1+\theta\phi_1(x)\phi_2(y)\geq 0 \text{ for all } x\in\mathbb{R},\ y\in(0,y_*]. \tag{3.1}$$

Trivially, if $\theta=0$ or $\phi_1(x)\equiv 0,\ x\in\mathbb{R}$, or $\phi_2(y)\equiv 0,\ y\in(0,y_*]$, then X_1 and Y_1 are independent. Choosing $\phi_1(x)=1-2F(x)$ and $\phi_2(y)=1-2G(y)$ for all $x\in\mathbb{R}$ and $y\in(0,y_*]$, leads to the well-known Farlie-Gumbel-Morgenstern (FGM) distribution.

By Proposition 1.1 in [19], we know that the two kernels are bounded. Precisely speaking, if (X_1,Y_1) follows a bivariate Sarmanov distribution, then there exist two positive constants b_1 and b_2 such that

$$|\phi_1(x)|\leq b_1 \text{ and } |\phi_2(y)|\leq b_2 \tag{3.2}$$

for all $x\in\mathbb{R}$ and $y\in(0,y_*]$.

By using Theorems 2.1 and 2.2, the main result of this section is presented below.

Corollary 3.1. *Consider the discrete-time risk model, where $\{(X_i,Y_i),\ i\geq 1\}$ are a sequence of i.i.d. random vectors with common Sarmanov distribution. Assume that the limit $\lim_{x\to\infty}\phi_1(x)=d_1$ exists. Under the conditions of Theorem 2.1, the following assertions hold with the new $K=(p_*-\theta d_1\mathbb{E}(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}}))/y_*^{\alpha-1}$.*

(1) If $0<y_*<1$, then the firest relation in (2.1) holds for each fixed $n\geq 1$, and second relation in (2.1) holds uniformly for all $n\geq 1$.

(2) If $y_*=1$, then relation (2.2) holds for each fixed $n\geq 1$.

(3) If $y_*>1$, then relation

$$\begin{aligned}&\mathbb{P}(M_n>x)\sim\mathbb{P}(S_n>x)\\&\sim\frac{K}{y_*^{(n-1)(\alpha-1)}}\prod_{i=1}^{n-1}\left(p_*\mathbb{E}\big(e^{\frac{\gamma X_1}{y_*^i}}\big)+\theta d_1\mathbb{E}\big(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}}\big)\mathbb{E}\Big(e^{\frac{\gamma\phi_1(X_1)}{y_*^i}}\Big)\right)l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*^n}}\end{aligned}$$

holds for each fixed $n\geq 1$.

We remark that the constant K in Corollary 3.1 is nonnegative. Indeed, by $\mathbb{E}\phi_2(Y_1)=0$ we have $p_*-\theta d_1\mathbb{E}(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}})=p_*(1+\theta d_1\phi_2(y_*))\geq 0$ due to (3.1).

4 Proofs

In this section we present the proofs of the main results.

Proof of Theorem 2.1. For claim (1), we firstly consider the first relation of (2.1). Denote

$$T_0=0,\ T_n\overset{\mathrm{d}}{=}\sum_{i=1}^{n}X_i\prod_{j=i}^{n}Y_j,\ n\geq 1,$$

then $T_n \overset{\mathrm{d}}{=} S_n,\ n \geq 1$, where $\overset{\mathrm{d}}{=}$ represents equality in distribution. Clearly, $\{T_n,\ n \geq 0\}$ satisfies the stochastic equation

$$T_0 = 0, \quad T_n = (T_{n-1} + X_n)Y_n,\ n \geq 1. \tag{4.1}$$

Similar stochastic recurrence equations can be found in [10]. Therefore, in order to prove the first relation of (2.1), we only need to verify the relation

$$\mathbb{P}(T_n > x) \sim K\mathbb{E}\big(e^{\gamma T_{n-1}}\big) l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \tag{4.2}$$

We proceed by induction on n. Trivially, relation (4.2) holds for $n = 1$ by $\mathbb{P}(T_1 > x) = \mathbb{P}(X_1 Y_1 > x)$ and Lemma 2 in [4], which implies that $\mathbb{E}(e^{\gamma T_1}) < \infty$ and $F_{T_1} \in \mathscr{L}(\gamma/y_*)$. Now we inductively assume that (4.2) holds for $n = m$ for some integer $m \geq 1$, hence $\mathbb{E}(e^{\gamma T_m}) < \infty$ and $F_{T_m} \in \mathscr{L}(\gamma/y_*)$. According to whether or not the events $(T_m > 0)$ and $(X_{m+1} > 0)$ happen we divide the tail probability $\mathbb{P}(T_m + X_{m+1} > x)$ into three parts as

$$\mathbb{P}(T_m + X_{m+1} > x) = \sum_{i=1}^{3} \mathbb{P}(T_m + X_{m+1} > x, \Omega_i) =: \sum_{i=1}^{3} I_i, \tag{4.3}$$

where $\Omega_1 = (T_m > 0, X_{m+1} > 0)$, $\Omega_2 = (T_m > 0, X_{m+1} \leq 0)$ and $\Omega_3 = (T_m \leq 0, X_{m+1} > 0)$. For any $0 < \epsilon < 1$ such that $0 < (1+\epsilon)y_* < 1$, we have that

$$I_1 = \left(\int_0^{\frac{x}{1+\epsilon}} + \int_{\frac{x}{1+\epsilon}}^{x} \right) \overline{F}(x-u)\mathbb{P}(T_m \in \mathrm{d}u) + \mathbb{P}(T_m > x)\overline{F}(0) =: \sum_{i=1}^{3} I_{1i}. \tag{4.4}$$

We firstly deal with I_{11}. By Theorem 1.5.6 (i) in [1], for any $\delta > 0$ and sufficiently large x, we have that

$$\frac{l(x-u)(x-u)^{\alpha-1}}{l(x)x^{\alpha-1}} \leq 2\left(\frac{x-u}{x}\right)^{\alpha-1-\delta} \leq 2\left(\left(\frac{\epsilon}{1+\epsilon}\right)^{\alpha-1-\delta} \vee 1\right)$$

with $0 \leq u \leq x/(1+\epsilon)$. Then, the dominated convergence theorem gives that

$$\begin{aligned} I_{11} &\sim \int_0^{\frac{x}{1+\epsilon}} l(x-u)(x-u)^{\alpha-1} e^{-\gamma(x-u)} \mathbb{P}(T_m \in \mathrm{d}u) \\ &\sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\mathbf{1}_{\{T_m>0\}}\big). \end{aligned} \tag{4.5}$$

As for I_{12} and I_{13}, by the induction assumption, we have that

$$\begin{aligned} I_{12} + I_{13} &\leq 2\mathbb{P}\Big(T_m > \frac{x}{1+\epsilon}\Big) \\ &\sim \frac{2K\mathbb{E}\big(e^{\gamma T_{m-1}}\big)}{(1+\epsilon)^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*(1+\epsilon)}}. \end{aligned} \tag{4.6}$$

Plugging (4.5) and (4.6) into (4.4) yields that

$$I_1 \sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\mathbf{1}_{\{T_m>0\}}\big). \tag{4.7}$$

As for I_3, according to the dominated convergence theorem and $F \in \mathscr{L}(\gamma)$, we obtain that

$$\begin{aligned} I_3 &= \int_{-\infty}^{0} \overline{F}(x-u)\mathbb{P}(T_m \in \mathrm{d}u) \\ &\sim \overline{F}(x)\int_{-\infty}^{0} e^{\gamma u}\mathbb{P}(T_m \in \mathrm{d}u). \end{aligned} \tag{4.8}$$

Similarly, again by the dominated convergence theorem, the fact that $F_{T_m} \in \mathscr{L}(\gamma/y_*)$ and the induction assumption, we have that

$$\begin{aligned} I_2 &\sim \mathbb{P}(T_m > x)\mathbb{E}\big(e^{\frac{\gamma X_1}{y_*}}\mathbf{1}_{\{X_1 \le 0\}}\big) \\ &\sim K\mathbb{E}\big(e^{\gamma T_{m-1}}\big)\mathbb{E}\big(e^{\frac{\gamma X_1}{y_*}}\mathbf{1}_{\{X_1 \le 0\}}\big)l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}. \end{aligned} \tag{4.9}$$

Thus, we derive from (4.3) and (4.7)–(4.9) that

$$\mathbb{P}(T_m + X_{m+1} > x) \sim \mathbb{E}\big(e^{\gamma T_m}\big)l(x)x^{\alpha-1}e^{-\gamma x}. \tag{4.10}$$

Therefore, by (4.10) and Lemma 2 in [4], the desired relation (4.2) holds for $n = m+1$, which implies that $\mathbb{E}\big(e^{\gamma T_{m+1}}\big) < \infty$.

For the second relation of (2.1), using the identity

$$M_n \stackrel{\mathrm{d}}{=} \bigvee_{k=0}^{n} T_k, \ n \ge 1,$$

and Theorem 2.1 in [14], we find that

$$M_n \stackrel{\mathrm{d}}{=} W_n, \ n \ge 1,$$

where $\{W_n, \ n \ge 1\}$ constitute a Markov chain defined by

$$W_0 = 0, \quad W_n = (W_{n-1} + X_n)_+ Y_n, \ n \ge 1. \tag{4.11}$$

Starting from (4.11) and proceeding along the same lines as above, we can obtain that

$$\mathbb{P}(W_n > x) \sim K\mathbb{E}\big(e^{\gamma W_{n-1}}\big)l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}},$$

which coincides with the second relation of (2.1). It ends the proof of Theorem 2.1 (1).

For claim (2), as explained in the above proof, we need to prove the relation

$$\mathbb{P}(T_n > x) \sim \frac{K^n\gamma^{n-1}(\Gamma(\alpha))^n}{\Gamma(n\alpha)}(l(x))^n x^{n\alpha-1}e^{-\gamma x}. \tag{4.12}$$

We proceed again by induction on n. Relation (4.12) trivially holds for $n = 1$ by taking into account Lemma 2 in [4]. Let us assume that relation (4.12) holds for $n = m$ for some integer $m \ge 1$, which implies that $F_{T_m} \in \mathscr{L}(\gamma)$. As we have done in (4.3), we split the tail probability $\mathbb{P}(T_m + X_{m+1} > x)$ into three parts, denoted by $I_1, \ I_2$ and I_3 as well. We firstly consider I_1. Construct two independent positive conditional r.v.s $X^c_{m+1} = (X_{m+1}|X_{m+1} > 0)$ and $T^c_m = (T_m|T_m > 0)$, whose tail distributions, by induction assumption, satisfy

$$\mathbb{P}(X^c_{m+1} > x) \sim \frac{1}{\overline{F}(0)} l(x) x^{\alpha-1} e^{-\gamma x},$$
$$\mathbb{P}(T^c_m > x) \sim \frac{K^m \gamma^{m-1} (\Gamma(\alpha))^m}{\mathbb{P}(T_m > 0)\Gamma(m\alpha)} (l(x))^m x^{m\alpha-1} e^{-\gamma x}.$$

Then, by Lemma 2.1 in [7], we have that

$$\begin{aligned} I_1 &= \mathbb{P}(T_m > 0)\mathbb{P}(X_{m+1} > 0)\mathbb{P}(T^c_m + X^c_{m+1} > x) \\ &\sim \frac{\gamma\Gamma(m\alpha)\Gamma(\alpha)}{\Gamma((m+1)\alpha)} \cdot \frac{K^m \gamma^{m-1} (\Gamma(\alpha))^m}{\Gamma(m\alpha)} (l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x} \\ &= \frac{K^m \gamma^m (\Gamma(\alpha))^{m+1}}{\Gamma((m+1)\alpha)} (l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x}. \end{aligned} \tag{4.13}$$

According to the dominated convergence theorem, we obtain from $F_{T_m} \in \mathscr{L}(\gamma)$ and the induction assumption that

$$\begin{aligned} I_2 &\sim \frac{K^m \gamma^{m-1} (\Gamma(\alpha))^m}{\Gamma(m\alpha)} (l(x))^m x^{m\alpha-1} e^{-\gamma x} \mathbb{E}\big(e^{\gamma X_1} \mathbf{1}_{\{X_1 \le 0\}}\big) \\ &= o(1)(l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x}. \end{aligned} \tag{4.14}$$

Analogously,

$$\begin{aligned} I_3 &\sim l(x) x^{\alpha-1} e^{-\gamma x} \mathbb{E}\big(e^{\gamma T_m} \mathbf{1}_{\{T_m \le 0\}}\big) \\ &= o(1)(l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x}. \end{aligned} \tag{4.15}$$

From (4.3) and (4.13)–(4.15) we obtain that

$$\mathbb{P}(T_m + X_{m+1} > x) \sim \frac{K^m \gamma^m (\Gamma(\alpha))^{m+1}}{\Gamma((m+1)\alpha)} (l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x},$$

which, by Lemma 2 in [4], leads to the desired relation (4.12) holding for $n = m+1$. This ends of Theorem 2.1 (2).

For claim (3), relation (2.3) can be derived by using a similar argument as that of Theorem 2.1 (1). Below, we show the asymptotic formulae for $\mathbb{P}(M_n > x)$ which is the refinement of relation (1.5). By (1.5), we only need to prove the relation

$$\mathbb{P}\Big(\sum_{i=1}^n y^i_* X_i > x\Big) \sim \frac{\prod_{i=1}^{n-1} \mathbb{E}\big(e^{\frac{\gamma X_1}{y^i_*}}\big)}{y_*^{n(\alpha-1)}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y^n_*}}. \tag{4.16}$$

As in the proof of Theorem 2.1 (1), relation (4.16) can be proved by proceeding with induction. $\square$

Proof of Theorem 2.2. We firstly prove the asymptotic relation (2.4) for the infinite-time ruin probability. For simplicity, for each $n \geq 0$, define nonnegative r.v.s

$$\xi_n = \sum_{i=n+1}^{\infty} y_*^i X_{i+}. \tag{4.17}$$

Clearly, by (1.2), for all $n \geq 1$,

$$0 \leq M_n \leq \xi_0 - \xi_n. \tag{4.18}$$

Notice that $\ln y_* < 0$ and $\mathbb{E}\ln(X_1 \vee 1) < \infty$, then by Theorem 1.6 in [17], M_n converges a.s. to a limit M_∞ as $n \to \infty$. Next, we show that $\mathbb{E}\big(e^{\gamma M_\infty}\big) < \infty$, motivated by an idea in [5], see also a related discussion in the proof of Theorem 3.2 in [11]. Let Z be a nonnegative r.v., independent of $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$, with tail distribution

$$\overline{F_Z}(x) \sim x^{2\alpha-1} e^{-\frac{\gamma x}{y_*}}.$$

Clearly, $\mathbb{E}(e^{\gamma Z}) < \infty$ due to $0 < y_* < 1$. Then, similarly to the proof of (4.10), we have that

$$\mathbb{P}((Z + X_{1+})y_* > x) \sim \frac{\mathbb{E}\big(e^{\gamma Z}\big)}{y_*^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}} = o(\overline{F_Z}(x)).$$

Hence, there exists some $x_0 > 0$ large enough such that for all $x \geq x_0$,

$$\mathbb{P}((Z + X_{1+})y_* > x) \leq \overline{F_Z}(x).$$

Construct a new nonnegative conditional r.v. $Z^c = (Z | Z > x_0)$. Clearly,

$$(Z^c + X_{1+})y_* \overset{\mathrm{d}}{\leq} Z^c, \tag{4.19}$$

where $\overset{\mathrm{d}}{\leq}$ means that for all $x \geq 0$,

$$\mathbb{P}((Z^c + X_{1+})y_* > x) \leq \mathbb{P}(Z > x | Z > x_0).$$

Similarly,

$$(Z^c + X_{2+})y_* \overset{\mathrm{d}}{\leq} Z^c,$$

which, together with (4.19), leads to

$$((Z^c + X_{2+})y_* + X_{1+})y_* \overset{\mathrm{d}}{\leq} Z^c.$$

Thus, by (4.18), $M_1 \leq y_* X_{1+} \overset{\mathrm{d}}{\leq} Z^c$ and $M_2 \leq y_* X_{1+} + y_*^2 X_{2+} \overset{\mathrm{d}}{\leq} Z^c$. Repeating these iterations we obtain $M_n \overset{\mathrm{d}}{\leq} Z^c$ for every $n \geq 1$. Letting $n \to \infty$ yields

$$M_\infty \leq \xi_0 \overset{\mathrm{d}}{\leq} Z^c,$$

where ξ_0 is defined in (4.17). This implies that

$$\mathbb{E}\big(e^{\gamma M_\infty}\big) \leq \mathbb{E}\big(e^{\gamma \xi_0}\big) < \infty, \tag{4.20}$$

because of $\overline{F_{Z^c}}(x) = \overline{F_Z}(x)/\overline{F_Z}(x_0)$ for all $x \geq x_0$ and $\mathbb{E}(e^{\gamma Z}) < \infty$. Then, by (4.20), the dominated convergence theorem, we have that

$$\lim_{n\to\infty} \mathbb{E}\big(e^{\gamma M_{n-1}}\big) = \mathbb{E}\big(e^{\gamma M_\infty}\big),$$

and by (4.20) and Jensen's inequality,

$$1 \leq \mathbb{E}\big(e^{\gamma \xi_{n-1}}\big) = \mathbb{E}\big(e^{\gamma \xi_0}\big)^{y_*^{n-1}} \leq \big(\mathbb{E}\big(e^{\gamma \xi_0}\big)\big)^{y_*^{n-1}} \to 1,\ n \to \infty.$$

For any $\epsilon > 0$ such that $0 < (1+\epsilon)y_* < 1$ and arbitrarily fixed $\bar{y} \in (\sqrt[3]{y_*}, 1)$ (implying $y_* < \bar{y}^3 < 1$), by the above two equations, we can choose a sufficiently large integer $n_0 \geq 3$ such that

$$\big|\mathbb{E}\big(e^{\gamma M_{n_0-1}}\big) - \mathbb{E}\big(e^{\gamma M_\infty}\big)\big| \leq \epsilon, \tag{4.21}$$

$$\mathbb{E}\left(e^{\gamma \xi_{n_0-1}}\right) \leq 1+\epsilon, \tag{4.22}$$

and

$$\sum_{i=n_0+1}^{\infty} \bar{y}^i < 1. \tag{4.23}$$

For the upper bound of (2.4),

$$\begin{aligned} \mathbb{P}(M_\infty > x) &\leq \mathbb{P}(M_{n_0} + \xi_{n_0} > x) \\ &= \left(\int_0^{\frac{x}{1+\epsilon}} + \int_{\frac{x}{1+\epsilon}}^{\infty}\right) \mathbb{P}(M_{n_0} > x-u)\mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &=: J_1 + J_2. \end{aligned} \tag{4.24}$$

As done in (4.5), by recalling the definition of $\psi_n(x)$ and using the second relation of (2.1) in Theorem 2.1, (4.21), the dominated convergence theorem and (4.22), we have that

$$\begin{aligned} J_1 &\sim \int_0^{\frac{x}{1+\epsilon}} \psi_{n_0}(x-u)\mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &\leq (1+\epsilon)\psi_\infty(x) \int_0^{\frac{x}{1+\epsilon}} \frac{l(x-u)(x-u)^{\alpha-1}}{l(x)x^{\alpha-1}} e^{\frac{\gamma u}{y_*}} \mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &\sim (1+\epsilon)\psi_\infty(x)\mathbb{E}\big(e^{\gamma \xi_{n_0-1}}\big) \\ &\leq (1+\epsilon)^2\psi_\infty(x). \end{aligned} \tag{4.25}$$

We next deal with J_2. For any $\delta > 0$, by (4.23) and $F \in \mathscr{R}_{-\infty}$, we have that for sufficiently large x,

$$\begin{aligned} \mathbb{P}(\xi_{n_0} > x) &\leq \mathbb{P}\Big(\sum_{i=n_0+1}^{\infty} y_*^i X_{i+} > x \sum_{i=n_0+1}^{\infty} \bar{y}^i\Big) \\ &\leq \sum_{i=n_0+1}^{\infty} \overline{F}\Big(\Big(\frac{\bar{y}}{y_*}\Big)^i x\Big) \\ &\leq 2\overline{F}\Big(\Big(\frac{\bar{y}}{y_*}\Big)^3 x\Big) \sum_{i=n_0+1}^{\infty} \Big(\frac{\bar{y}}{y_*}\Big)^{-(i-3)\delta} \\ &\sim C_0 l(x)x^{\alpha-1}e^{-\gamma\left(\frac{\bar{y}}{y_*}\right)^3 x}, \end{aligned}$$

where $C_0 = 2\left(\frac{\bar{y}}{y_*}\right)^{3(\alpha-1)} \sum_{i=n_0+1}^{\infty} \left(\frac{\bar{y}}{y_*}\right)^{-(i-3)\delta} < \infty$, and in the third step we used $\overline{F}(y)/\overline{F}(x) \le (1+\epsilon)(y/x)^{-\delta}$ for any $\epsilon > 0$, $\delta > 0$ and sufficiently large $y \ge x$, see Theorem 1.2.2 in [6] or (2.1) in [15]. This yields that

$$\begin{aligned} J_2 &\le \mathbb{P}\Big(\xi_{n_0} > \frac{x}{1+\epsilon}\Big) \\ &\lesssim \frac{C_0}{(1+\epsilon)^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma}{1+\epsilon}\left(\frac{\bar{y}}{y_*}\right)^3 x} \\ &= o(1) l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}, \end{aligned} \tag{4.26}$$

where the last step holds by taking account of the fact

$$\frac{1}{1+\epsilon}\left(\frac{\bar{y}}{y_*}\right)^3 = \frac{1}{y_*} \cdot \frac{1}{(1+\epsilon)y_*} \cdot \frac{\bar{y}^3}{y_*} > \frac{1}{y_*}.$$

Plugging (4.25) and (4.26) into (4.24), we obtain that

$$\mathbb{P}(M_\infty > x) \lesssim (1+\epsilon)^2 \psi_\infty(x).$$

For the lower bound of (2.4), by the second relation of (2.1) in Theorem 2.1 and (4.21), we derive that

$$\mathbb{P}(M_\infty > x) \ge \mathbb{P}(M_{n_0} > x) \sim \psi_{n_0}(x) \ge (1-\epsilon)\psi_\infty(x).$$

Therefore, the desired relation (2.4) follows by the arbitrariness of $\epsilon > 0$. It claims the first part of Theorem 2.2.

Now we prove that the second relation of (2.1) holds uniformly for all $n \ge 1$. For any positive x and any $N \ge 2$ we have that

$$\begin{aligned} \sup_{n\ge1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)} &\le \left(\bigvee_{n=1}^{N-1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)}\right) \vee \left(\bigvee_{n=N}^{\infty} \frac{\mathbb{P}(M_\infty > x)}{\psi_n(x)}\right) \\ &\le \left(\bigvee_{n=1}^{N-1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)}\right) \vee \left(\frac{\mathbb{P}(M_\infty > x)}{\psi_\infty(x)} \cdot \frac{\mathbb{E}\big(e^{\gamma M_\infty}\big)}{\mathbb{E}\big(e^{\gamma M_{N-1}}\big)}\right). \end{aligned}$$

According to the second relation of (2.1) and (2.4), we obtain that

$$\limsup_{x\to\infty} \sup_{n\ge1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)} \le 1 \vee \frac{\mathbb{E}\big(e^{\gamma M_\infty}\big)}{\mathbb{E}\big(e^{\gamma M_{N-1}}\big)}.$$

The upper estimate follows now from the last inequality by letting $N \uparrow \infty$ and (4).

Similarly, for the lower bound,

$$\begin{aligned} \inf_{n\ge1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)} &\ge \left(\bigwedge_{n=1}^{N-1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)}\right) \wedge \left(\bigwedge_{n=N}^{\infty} \frac{\mathbb{P}(M_N > x)}{\psi_n(x)}\right) \\ &\ge \left(\bigwedge_{n=1}^{N-1} \frac{\mathbb{P}(M_n > x)}{\psi_n(x)}\right) \wedge \left(\frac{\mathbb{P}(M_N > x)}{\psi_N(x)} \cdot \frac{\mathbb{E}\big(e^{\gamma M_{N-1}}\big)}{\mathbb{E}\big(e^{\gamma M_\infty}\big)}\right). \end{aligned}$$

Using the second relation of (2.1), we have that

$$\liminf_{x\to\infty}\inf_{n\geq 1}\frac{\mathbb{P}(M_n>x)}{\psi_n(x)}\geq 1\wedge\frac{\mathbb{E}\big(e^{\gamma M_{N-1}}\big)}{\mathbb{E}\big(e^{\gamma M_\infty}\big)}.$$

Therefore, the lower bound is derived also by letting $N\uparrow\infty$ and (4). This completes the proof of Theorem 2.2. □

Proof of Corollary 3.1. Clearly, the recursive equations (4.1), (4.11) and the identities $S_n\overset{\mathrm{d}}{=}T_n$, $M_n\overset{\mathrm{d}}{=}W_n$ for every $n\geq 1$ still hold, due to the i.i.d. assumption for the sequence $\{(X_i,Y_i),\ i\geq 1\}$. For each fixed $n\geq 1$, the tail probability $\mathbb{P}(T_n>x)$ can be separated into four parts as

$$\begin{aligned}\mathbb{P}(T_n>x)&=\mathbb{P}((T_{n-1}+X_n)Y_n>x)\\&=(1+\theta b_1b_2)\mathbb{P}((T_{n-1}+X^*)Y^*>x)-\theta b_1b_2\mathbb{P}((T_{n-1}+\widetilde{X}^*)Y^*>x)\\&\quad-\theta b_1b_2\mathbb{P}((T_{n-1}+X^*)\widetilde{Y}^*>x)+\theta b_1b_2\mathbb{P}((T_{n-1}+\widetilde{X}^*)\widetilde{Y}^*>x)\\&=:(1+\theta b_1b_2)K_{1n}-\theta b_1b_2K_{2n}-\theta b_1b_2K_{3n}+\theta b_1b_2K_{4n},\end{aligned}\tag{4.27}$$

where $X^*,\ \widetilde{X}^*,\ Y^*$ and $\widetilde{Y}^*$ are four independent r.v.s, independent of $\{(X_i,Y_i),\ i\geq 1\}$, with distributions $F,\ \widetilde{F},\ G$ and $\widetilde{G}$, respectively, defined by

$$\widetilde{F}(\mathrm{d}x)=\left(1-\frac{\phi_1(x)}{b_1}\right)F(\mathrm{d}x)\ \text{ and }\ \widetilde{G}(\mathrm{d}y)=\left(1-\frac{\phi_2(y)}{b_2}\right)G(\mathrm{d}y),\ x\in\mathbb{R},\ y\in(0,y_*).\tag{4.28}$$

Here, b_1 and b_2 are two large positive constants defined in (3.2) such that $|\phi_1(x)|\leq b_1-1,\ |\phi_2(y)|\leq b_2-1$ for all $x\in\mathbb{R}$ and $y\in(0,y_*]$. Since $\lim_{x\to\infty}\phi_1(x)=d_1$ and by (4.28) we have that

$$\overline{\widetilde{F}}(x)=\int_x^\infty\left(1-\frac{\phi_1(u)}{b_1}\right)F(\mathrm{d}u)\sim\left(1-\frac{d_1}{b_1}\right)\overline{F}(x),$$

$\widetilde{Y}^*$ has the same upper endpoint y_* as that of Y_1, and

$$\mathbb{P}(\widetilde{Y}^*=y_*)=p_*+\frac{1}{b_1}\mathbb{E}\big(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}}\big)\geq 0,$$

for some large b_1. Hence, when $n=1$, by Lemma 2 in [4],

$$\begin{cases}K_{11}\sim\frac{p_*}{y_*^{\alpha-1}}l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}},\\K_{21}\sim(1-\frac{d_1}{b_1})\frac{p_*}{y_*^{\alpha-1}}l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}},\\K_{31}\sim\big(p_*+\frac{1}{b_2}\mathbb{E}\big(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}}\big)\big)\frac{1}{y_*^{\alpha-1}}l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}},\\K_{41}\sim(1-\frac{d_1}{b_1})\big(p_*+\frac{1}{b_2}\mathbb{E}\big(\phi_2(Y_1)\mathbf{1}_{\{0<Y_1<y_*\}}\big)\big)\frac{1}{y_*^{\alpha-1}}l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}.\end{cases}\tag{4.29}$$

Plugging (4.29) into (4.27) yields

$$\mathbb{P}(S_1>x)=\mathbb{P}(T_1>x)\sim Kl(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}.$$

When $0<y_*<1$, as in the proof of Theorem 2.1 (1), proceeding with induction according to (4.27) leads to the desired relation (2.1).

By the same argument, the results in the other cases can also be derived. □

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